Risk Flow Patterns

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Motivation

Cash flow patterns...

- ... help to determine "what part of our reserves become payable between k and ℓ years from now?"
  - liquidity mgmt, ALM, duration matching, discounting, IFRS 4 & 17
- ... are considered as characteristics of lines of business
  - benchmarking, regulatory use (e.g. FINMA SST patterns)
- ... have nice properties:
  - volume-independent, transform naturally upon change in time granularity.
- ... are a natural set of parameters for the deterministic chain ladder model.

Can we have something similar for the risk ???

The answer is "yes"!

Summary of Results

In this talk, we will introduce **"risk flow patterns"** that

- ... help determine "what part of the insurance risk materializes between k and ℓ years from now?"
  - cost of capital, SST, Solvency II, IFRS 17
- ... may be considered as characteristics of lines of business
  - benchmarking, regulatory use
- ... have nice properties:
  - volume-independent, transform naturally upon change in time granularity
- ... are a natural set of parameters for Mack's stochastic chain ladder model.
- ... allow for **radical simplifications** of the well-known prediction error formulae originally derived by Mack, Merz-Wüthrich and others.

For proofs, we refer to Röhr (2016).

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Chain Ladder Method

$C_{i,j} > 0$ is the cumulative paid or incurred loss from accident period i at development step $j \in \{0, \ldots, J\}$.

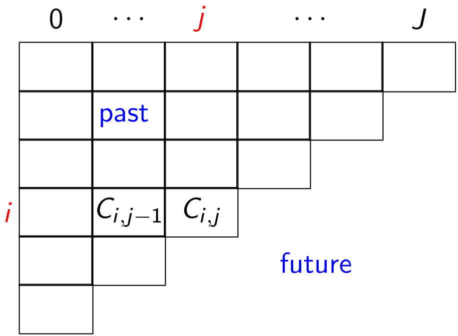
The known part of these form a loss development triangle.

Ultimates at $j = J$.

Link ratios $f_{i,j} = C_{i,j}/C_{i,j-1}$.

Chain Ladder Principle: predict future values by

$$\hat{C}_{i,j} := \begin{cases} C_{i,j} & \text{if known,} \\ \hat{f}_j \hat{C}_{i,j-1} & \text{else.} \end{cases}$$



Development Factor Estimator

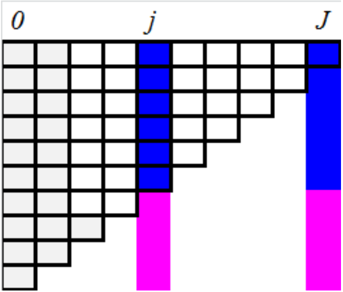
Use $\hat{f}_j := C_{\mathcal{I}_j,j}/C_{\mathcal{I}_j,j-1}$ where
$\mathcal{I}_j := \{i | C_{i,j} \text{ known today}\}$,
$C_{\mathcal{H},j} := \sum_{i\in\mathcal{H}} C_{i,j}$.

$\hat{C} := \hat{C}_{\mathcal{I}_0,J}$ (the predicted ultimate loss for all accident periods combined).

Influence factors

A loss development triangle, bottom/right filled with CL predictors.

$$\hat{q}_j := \frac{\text{Future}_j}{\text{Past}_j + \text{Future}_j}$$



The proportion of the ultimate $\hat{C}$ that is affected by $\hat{f}_j$ is just $\hat{q}_j$.

"Geometrically", $\hat{q}_j = 40\%$ in this picture.
We call the $\hat{q}_j$ the influence factors. We have $0 = \hat{q}_0 \leq \hat{q}_1 \leq \ldots \leq \hat{q}_J < 1$.

For convenience of notation, we set $\hat{q}_j := 0$ for $j \leq 0$.

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Mack's Stochastic Model (1993) - single accident period

A chain ladder process is a discrete-time, real-valued stochastic process $\{X_j > 0\}_{j \geq 0}$, such that for each $j > 0$

$$E\left[X_j|\mathcal{F}_{j-1}\right] = f_j\, X_{j-1},$$
$$V\left[X_j|\mathcal{F}_{j-1}\right] = \phi_j\, X_{j-1}$$

with parameters $f_j > 0$ (development factors) and $\phi_j \geq 0$, and where $\mathcal{F}_{j-1}$ is the σ-algebra generated by $X_0, \ldots, X_{j-1}$.

- Accident periods in loss triangle stoch. indep., but same parameters
- Standard estimators from loss triangle, $1 \leq j \leq J$:

$$\hat{f}_j := \frac{C_{\mathcal{I}_j,j}}{C_{\mathcal{I}_j,j-1}}, \qquad \hat{\phi}_j := \frac{\sum_{i\in\mathcal{I}_j} C_{i,j-1}\left(f_{i,j} - \hat{f}_j\right)^2}{-1 + \sum_{i\in\mathcal{I}_j} 1}$$

- Let's iterate the recursive properties above...

Cash Flow Patterns and Risk Flow Patterns

Proposition see Röhr (2016)

Assume the chain ladder process $\{X_j\}_{j\geq 0}$ becomes constant after step J (i.e. $f_j = 1$ and $\phi_j = 0$ for $j > J$). Then

$$
\begin{aligned}
E[X_J - X_{j-1}|\mathcal{F}_{j-1}] &= (\pi_j + \pi_{j+1} + \ldots + \pi_J)E[X_J|\mathcal{F}_{j-1}] \\
V[X_J|\mathcal{F}_{j-1}] &= (\rho_j + \rho_{j+1} + \ldots + \rho_J)E[X_J|\mathcal{F}_{j-1}]
\end{aligned}
$$

where $\Pi_j := f_{j+1} \cdot \ldots \cdot f_J$, $\pi_j := \Pi_j^{-1} - \Pi_{j-1}^{-1}$ and $\rho_j := \Pi_j \phi_j / f_j$.

- It pays to express everything in terms of the expected ultimate.
- The π_j are known as the cash flow pattern.
- We call the ρ_j the risk flow pattern.
- The ρ_j have the same dimension as the X_j.
- Get estimators $\hat{\pi}_j$, $\hat{\rho}_j$ via $\hat{f}_j$, $\hat{\phi}_j$.
- Both patterns behave nicely upon change of time granularity.

Example (Mack)

4370	6293	10292	12460	13660	14307
2701	5291	7162	8945	9338	9780
4483	6729	10074	11142	11971	12538
3254	5804	8351	9874	10608	11111
8010	12118	18028	21315	22901	23986
5582	8864	13187	15592	16752	17546

$\hat{f}_j =$	1.588	1.488	1.182	1.074	1.047
$\hat{q}_j =$	20%	47%	59%	73%	84%
$\hat{\pi}_j =$	18.7%	24.6%	13.7%	6.6%	4.5%
$\hat{\rho}_j =$	209.1	73.6	47.0	13.9	3.9

From data, get. . .

- link ratios $f_{i,j}$;
- estimator $\hat{f}_j$ for f_j;
- predicted loss development $\hat{C}_{i,j}$;
- influence factors $\hat{q}_j$
- cash flow pattern $\hat{\pi}_j$ (N.B. $\hat{\pi}_0 = 31.8\%$ not shown here)
- risk flow pattern $\hat{\rho}_j$

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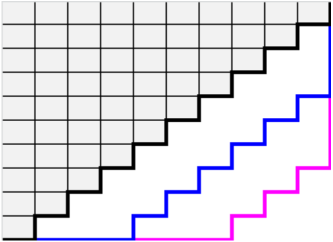


Development "Horizons":

black 0,

blue k,

magenta ℓ

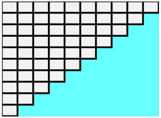
accounting periods from today

Development between k and ℓ periods from today

$$\text{cash flow} \quad = \quad \hat{C} \sum_{j=1}^{J} \hat{\pi}_j \left(\hat{q}_{j-k} - \hat{q}_{j-\ell} \right)$$

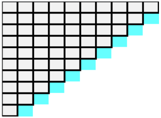
$$\text{(cond.) MSEP of loss dev. res. 0} \quad \approx \quad \hat{C} \sum_{j=1}^{J} \hat{\rho}_j \left(\frac{1}{1 - \hat{q}_{j-k}} - \frac{1}{1 - \hat{q}_{j-\ell}} \right)$$

MSEP Formulae Based on Mack's Stochastic Model



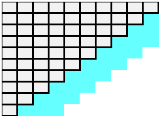
Mack 1993

$k = 0$ (today) $\longrightarrow$ $\ell = J$ (ultimate)



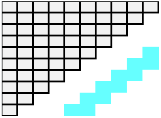
Merz/Wüthrich 2008

$k = 0$ (today) $\longrightarrow$ $\ell = 1$ period from now



Diers et al. 2016

$k = 0$ (today) $\longrightarrow$ ℓ periods from now



Our version (also Merz/Wüthrich 2014, Gisler 2016)

k periods from now $\longrightarrow$ ℓ periods from now

Mack (1993)

$$\widehat{mse}(\hat{R}_i) = \hat{C}_{iI}^2 \sum_{k=I+1-i}^{I-1} \frac{\hat{\sigma}_k^2}{\hat{f}_k^2} \left(\frac{1}{\hat{C}_{ik}} + \frac{1}{\sum_{j=1}^{I-k} C_{jk}} \right)$$

$$\widehat{mse}(\hat{R}) = \sum_{i=2}^{I} \left\{ (\text{s.e.}\,(\hat{R}_i))^2 + \hat{C}_{iI} \left(\sum_{j=i+1}^{I} \hat{C}_{jI} \right) \sum_{k=I+1-i}^{I-1} \frac{2\hat{\sigma}_k^2 / \hat{f}_k^2}{\sum_{n=1}^{I-k} C_{nk}} \right\}$$

Our version (algebraically identical) $k = 0,\ \ell = J$

$$\text{MSEP} \approx \hat{C} \sum_{j=1}^{J} \hat{\rho}_j \left(\frac{1}{1 - \hat{q}_j} - 1 \right)$$

Comparison with Merz/Wüthrich's Formula

Merz/Wüthrich (2008), see Bühlmann et al. (2009)

$$\widehat{\mathrm{msep}}_{\widehat{\mathrm{CDR}}_i(I+1)\,|\,\mathcal{D}_I}(0) \tag{4.19}$$
$$= \left(\widehat{C}_{i,J}^{CL}\right)^2 \left[\frac{\sigma_{I-i}^2 / \left(\widehat{F}_{I-i}^{(I)}\right)^2}{C_{i,I-i}} + \frac{\sigma_{I-i}^2 / \left(\widehat{F}_{I-i}^{(I)}\right)^2}{S_{I-i}^{[i-1]}} + \sum_{j=I-i+1}^{J-1} \frac{C_{I-j,j}}{S_j^{[I-j]}} \frac{\sigma_j^2 / \left(\widehat{F}_j^{(I)}\right)^2}{S_j^{[I-j-1]}} \right]$$

$$\widehat{\mathrm{msep}}_{\sum_{i=I-J+1}^{I} \widehat{\mathrm{CDR}}_i(I+1)\,|\,\mathcal{D}_I}(0) = \sum_{i=I-J+1}^{I} \widehat{\mathrm{msep}}_{\widehat{\mathrm{CDR}}_i(I+1)\,|\,\mathcal{D}_I}(0) \tag{4.20}$$
$$+ 2 \sum_{I-J+1 \le i < k \le I} \widehat{C}_{i,J}^{CL} \widehat{C}_{k,J}^{CL} \left[\frac{\sigma_{I-i}^2 / \left(\widehat{F}_{I-i}^{(I)}\right)^2}{S_{I-i}^{[i-1]}} + \sum_{j=I-i+1}^{J-1} \frac{C_{I-j,j}}{S_j^{[I-j]}} \frac{\sigma_j^2 / \left(\widehat{F}_j^{(I)}\right)^2}{S_j^{[I-j-1]}} \right].$$

Our version (algebraically identical) $k = 0$, $\ell = 1$

$$\mathrm{MSEP} \approx \hat{C} \sum_{j=1}^{J} \hat{\rho}_j \left(\frac{1}{1-\hat{q}_j} - \frac{1}{1-\hat{q}_{j-1}} \right)$$

Comparison with Merz/Wüthrich's "Full Picture" Formula

Merz/Wüthrich (2014)

$$
\begin{aligned}
\varrho^{(I)}_{i,I+k+1} &= \widehat{\mathbb{E}}\left[\mathrm{msep}^{\mathrm{MW}}_{\mathrm{CDR}_{i,I+k+1}|\mathcal{D}_{I+k}}(0)\,\middle|\,\mathcal{D}_I\right] \\
&= \left(\widehat{C}^{CL(I)}_{i,J}\right)^2 \frac{s^2_{I-i+k}}{\left(\widehat{f}^{CL(I)}_{I-i+k}\right)^2}\left[\frac{1}{\widehat{C}^{CL(I)}_{i,I-i+k}} + \prod_{m=1}^{k}\left(1-\alpha^{(I)}_{I-i+m}\right)\frac{1}{\sum_{\ell=1}^{i-k-1} C_{\ell,I-i+k}}\right] \qquad (1.4) \\
&\quad + \left(\widehat{C}^{CL(I)}_{i,J}\right)^2 \sum_{j=I-i+k+1}^{J-1} \frac{s^2_j}{\left(\widehat{f}^{CL(I)}_j\right)^2}\left[\alpha^{(I)}_{j-k}\prod_{m=0}^{k-1}\left(1-\alpha^{(I)}_{j-m}\right)\frac{1}{\sum_{\ell=1}^{I-j-1} C_{\ell,j}}\right].
\end{aligned}
$$

$$
\begin{aligned}
\varrho^{(I)}_{I+k+1} &= \widehat{\mathbb{E}}\left[\mathrm{msep}^{\mathrm{MW}}_{\sum_{i=I-J+k+1}^{I}\mathrm{CDR}_{i,I+k+1}|\mathcal{D}_{I+k}}(0)\,\middle|\,\mathcal{D}_I\right] = \sum_{i=I-J+k+1}^{I} \varrho^{(I)}_{i,I+k+1} \qquad (2.4) \\
&+ 2 \sum_{I-J+k+1\le i<n\le I} \widehat{C}^{CL(I)}_{i,J}\widehat{C}^{CL(I)}_{n,J} \frac{s^2_{I-i+k}}{\left(\widehat{f}^{CL(I)}_{I-i+k}\right)^2} \prod_{m=1}^{k}\left(1-\alpha^{(I)}_{I-i+m}\right)\frac{1}{\sum_{\ell=1}^{i-k-1} C_{\ell,I-i+k}} \\
&+ 2 \sum_{I-J+k+1\le i<n\le I} \widehat{C}^{CL(I)}_{i,J}\widehat{C}^{CL(I)}_{n,J} \sum_{j=I-i+k+1}^{J-1} \frac{s^2_j}{\left(\widehat{f}^{CL(I)}_j\right)^2}\left[\alpha^{(I)}_{j-k}\prod_{m=0}^{k-1}\left(1-\alpha^{(I)}_{j-m}\right)\frac{1}{\sum_{\ell=1}^{I-j-1} C_{\ell,j}}\right].
\end{aligned}
$$

Our version (algebraically identical) $\ell = k + 1$

$$
\mathrm{MSEP} \approx \hat{C} \sum_{j=1}^{J} \hat{\rho}_j \left(\frac{1}{1-\hat{q}_{j-k}} - \frac{1}{1-\hat{q}_{j-(k+1)}} \right)
$$

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Interpreting the MSEP formula

$$\hat{C} \quad \sum_j \hat{\rho}_j \left(\frac{1}{1-\hat{q}_{j-k}} - \frac{1}{1-\hat{q}_{j-\ell}} \right)$$

Volume   Risk Flow Pattern   Triangle Geometry

- All volume dependence is captured by the total predicted ultimate loss $\hat{C}$!
- Risk flow pattern ρ_j: volume-independent, characteristic of the line of business;
- Influence factors $\hat{q}_j$ depend on data, but have nothing to do with the parameters governing the stochastic model, and may often be approximated by "geometry". For example,

$$\hat{q}_j \approx \frac{j}{J+1}$$

may be a reasonable average value for roughly constant business volume.

Application: Regulatory Solvency Models

- Current standard regulatory reserve risk models use

$$\text{Reserve Risk} = \text{Reserve} \cdot \alpha, \quad (\text{e.g. } \alpha = 8\%),$$

  - where α is company-individual (hence, non-standard), or
  - the risk does not diversify with volume.

- Our MSEP formula opens up the possibility to use

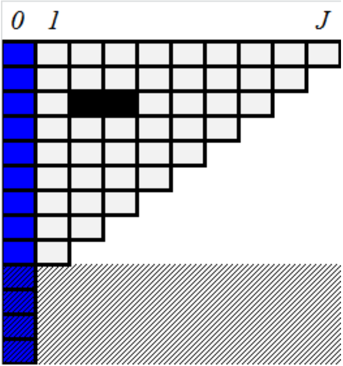
$$\text{Reserve Risk} = \sqrt{\text{Ultimate} \cdot \beta}, \quad (\text{e.g. } \beta = 250\,000 \text{ CHF}),$$

  which does diversify with volume, and where

  - the result is "fully Merz/Wüthrich compatible";
  - $\beta = \sum_j \hat{\rho}_j (1/(1 - \hat{q}_{j-k}) - 1/(1 - \hat{q}_{j-\ell}))$ is justifiably "entity-independent":
  - the risk flow pattern ρ_j may be prescribed per line of business and
  - the influence factors $\hat{q}_j$ may also be prescribed, based on industry averages, or "geometrically".

Not much complexity is added to our MSEP formula by

- allowing "ragged" triangle data; e.g., taking premium (or other volume measure) as first column (blue area) $\longrightarrow$ integrated view of reserve and premium risk (see also Diers et al. (2016));
- measuring the prediction error only for a subportfolio (shaded area) — splitting off, for example, the premium risk (or the risk adjustment for the remaining coverage under IFRS 17);



- dealing with unreliable, "deleted" data (black entries).

See Röhr (2016) for details.

Application: Aggregate Statistics

From cash flow pattern, get aggregate statistics

- duration
- discount factors

On the risk side, a statistic of interest may be the "total risk flow" $\sum_j \rho_j$.
NB: it only captures risk *after* the end of the first development step.

If the first development step is the first year of loss development, then typical values for the total (reserve) risk flow are:

- order of CHF 10^4: light short tail business
- order of CHF 10^5: medium to long tail business
- order of CHF 10^6: medium or long tail business with large risks

If the first development step is the premium (see previous slide), "premium risk" is included in the risk flow pattern, and these values become considerably larger.

- Continuous time t version of the equations in the Proposition:

$$E[X_J - X_t|\mathcal{F}_t] = \left(\int_t^J \pi_s ds\right) E[X_J|\mathcal{F}_t]$$
$$V[X_J|\mathcal{F}_t] = \left(\int_t^J \rho_s ds\right) E[X_J|\mathcal{F}_t]$$

- Among Itô diffusions, this is solved by the solution X_t to the SDE

$$dX_t = \beta_t X_t dt + \sqrt{\phi_t X_t} dB_t$$

  with deterministic, time-dependent functions $\phi_t \geq 0$ and β_t, where $\pi_t = \beta_t e^{-\int_t^J \beta_s ds}$, $\rho_t = \phi_t e^{\int_t^J \beta_s ds}$ and B_t Brownian motion.
- "Square root processes"; a related, well-known special case of these is the Cox-Ingersoll-Ross process r_t defined by

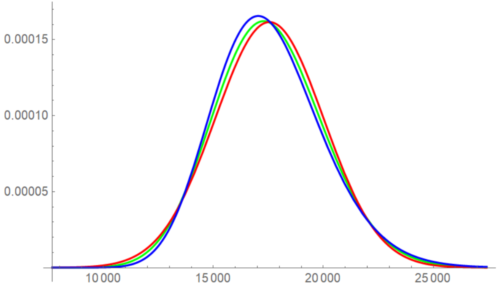
$$dr_t = a\,(b - r_t)dt + c\sqrt{r_t}dB_t$$

- Probability density of $X_J|X_t$ is well-known in closed form; fat point at 0 of mass e^{-2/z_t}, where $z_t = (\int_t^J \rho_s ds)/E[X_J|X_t] = (\text{coeff. of var.})^2$.

Application: Continuous Stochastic Chain Ladder Model

Example: last accident period in above triangle, distr. of ultimate $C_{i,J}$:

$$\hat{C}_{i,J} = 17546, \quad \sum_{j=1}^{J} \hat{\rho}_j = 347.5, \quad \text{coefficient of variation} = 14.1\%$$



- green "true" prob. density (regular part)
- red normal distribution
- blue lognormal distribution

- Fat point 0 of true distribution has mass $2 * 10^{-44}$, negligible
- True distr. has heavier tail than normal, but lighter than lognormal

Conclusion

Risk flow patterns...

- ... re-parameterize Mack's stochastic chain ladder model
- ... allow calibration on one portfolio and application to another portfolio (benchmarking)
- ... may be considered as invariants of lines of business
- ... also work in continuous time

The simplified MSEP formulae ...

- ... have interpretable parts
- ... clearly separate the influence of volume, risk parameters and historic portfolio structure on the risk estimation
- ... provide a starting point for modifications and adaptations (e.g., the "geometric" estimates of the $\hat{q}_j$)
- ... last not least, are easier to remember and to program

References

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